Application

We consider two-dimensional flow about a thin profile and let the values at the surface of the tangential and normal velocities of a lifting flow (or the normal and tangential velocities of a nonlifting flow) be denoted by $u(x)$ and $w(x)$, respectively. These quantities are approximately related by

$$\oint_{-1}^{1} u(y)(y - x)^{-1}dy = \pi w(x) \tag{11}$$

Applying the quadrature formula (3) to the Cauchy integral in Eq. (11), we immediately obtain the discretized counterpart

$$\sum_{n=1}^{N} a_n u(x_n)(x_n - X_m)^{-1} = \pi w(X_m) \qquad m = 1,2, \ldots \tag{12}$$

If $u(x)/W(x)$ is a polynomial of degree $\leqq 2N$, the values $u(x_n)$ and $w(X_m)$ are exact values of functions $u(x)$ and $w(x)$ related by Eq. (11).

We illustrate this in Fig. 1 in the case of $W(x) = 1$, i.e., in a case where the tangential velocity itself (or the normal velocity of a nonlifting flow) is a polynomial. The two solid curves represent the function $u(x) = 1 - [(1 + x)/2]^8$ and the corresponding w-function, which is defined by Eq. (11). We have chosen $N = 4$. Therefore, and as the order of the polynomial considered is equal to the maximum order $2N = 8$, the relation (12) is exactly valid. The points $[X_m,w(X_m)]$ and $[x_n,u(x_n)]$ must accordingly lie on the solid curves, as is shown by the circles in the figure.

It is interesting to compare the present discretized method with the vortex-lattice method. The latter employs as integration points (vortex points) and singularity locations (control points) the $\frac{1}{4}$ and $\frac{3}{4}$ points on N equal subintervals of the chord. It is generally known that these locations yield surprisingly good results in ordinary cases. In the present example, by using w values agreeing with the w curve, they yield u values which have been marked by squares in the figure.

A similar example but for constant loading has been treated by James[6] by the vortex-lattice method. His results deviate, however, rather much (about 9% for $N = 20$) from the true solution. As the u function considered by James is nonzero at the trailing edge, while the present one vanishes there, we are tempted to conclude that the vortex-lattice theory is unsuitable for functions which do not satisfy the Kutta-Joukowsky condition.

In the ordinary case of a lifting flow with constant or polynomial downwash, it is appropriate to choose $W(x) = [(1 - x)/(1 + x)^{1/2}]$, and for a slender or a finite wing it is also relevant to consider the weight functions $(1 - x^2)^{-1/2}$ and $(1 - x^2)^{1/2}$. Examples for these are not given here, for the formula (3) is then equivalent with the formula of Brakhage, and applications of this were shown in Ref. 7. Borja and Brakhage[7] achieved such a transformation of the basic integral relation that the quadrature formula can be applied also for a finite wing for chord-wise and span-wise integration. In a similar way, Borja[8] treated recently even the problem of an oscillating wing in incompressible flow.

We finally note that, if it is required to evaluate a weighted integral

$$L = \int_{-1}^{1} H(x)u(x)dx \tag{13}$$

as is often the case in aerodynamics, this can be done in a straightforward way by means of the Gaussian formula

$$L \approx \sum_{n=1}^{N} H(x_n)a_n u(x_n) \tag{14}$$

This does not require knowledge of a_n as the products $a_n u(x_n)$ can be solved from Eq. (12). The formula (14) yields exact values for L if $H(x)$ is a polynomial and if the degree of this plus the degree of $u(x)/W(x)$ is less than $2N$. Hence, the expressions (12) and (14) may be said to represent a generalization of the $\frac{3}{4}$ chord point formula of Pistolesi,[9] but not only to any number of vortices and control points but also to loadings characterized by an arbitrary weight function $W(x)$.

Conclusions

It has been demonstrated that the Gaussian quadrature formulas can be applied in the ordinary way even to a Cauchy integral if the singularity is located at any of certain appropriate points. Such points have been defined for integrands containing an arbitrary weight function and a regular factor. The formulas are exactly valid for polynomial factors of a degree equal to twice the number of integration points.

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Wave Propagation in Three-Layered Plates

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CONSIDER a traction-free, infinitely long plate made of three layers of different isotropic materials of different thickness (see Fig. 1). The material constants and geometry pertinent to layers 1, 2, or 3 will be designated by the superscripts (1), (2), or (3), respectively.

The solution of the Navier equations of motion for the ith

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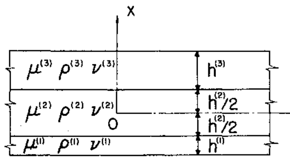

Fig. 1 Geometry of the three-layered plate.

($i = 1,2,3$) layer of the plate, may be written as[1]

$$u_x^{(i)} = k\{-A_1^{(i)}\epsilon_d^{(i)}\beta^{(i)}Z_1(k\beta^{(i)}x) - B_1^{(i)}\beta^{(i)}W_1(k\beta^{(i)}x) - A_2^{(i)}Z_0(k\alpha^{(i)}x) - B_2^{(i)}W_0(k\alpha^{(i)}x\} \cos(kz - \omega t)$$

$$u_y^{(i)} = -\{A_3^{(i)}Z_0(k\alpha^{(i)}x) + B_3^{(i)}W_0(k\alpha^{(i)}x\}k^2 \times (\epsilon_s\alpha^{(i)2} + 1) \sin(kz - \omega t)$$

$$u_z^{(i)} = k\{-A_1^{(i)}Z_0(k\beta^{(i)}x) - B_1^{(i)}W_0(k\beta^{(i)}x) - A_2^{(i)}\epsilon_s^{(i)}k\alpha^{(i)}Z_1(k\alpha^{(i)}x) - B_2^{(i)}\alpha^{(i)}W_1(k\alpha^{(i)}x)\} \times \sin(kz - \omega t) \quad (1)$$

where

$$\epsilon_j^{(i)} = \begin{cases} 1 \text{ if } v^2 > v_j^{(i)2} \\ -1 \text{ if } v^2 < v_j^{(i)2}\ (j = s,d) \end{cases} \quad (2)$$

$$(\alpha^i)^2 = \left|\frac{\rho^{(i)}\omega^2}{\mu^{(i)}k^2} - 1\right|,\ \beta^{(i)})^2 = \left|\frac{\rho^{(i)}\omega^2}{(\lambda^{(i)} + 2\mu^{(i)})k^2} - 1\right| \quad (3)$$

and

$$Z_0(k\gamma_j x) = \begin{cases}\cos(k\gamma_j x) \\ \exp(k\gamma_j x)\end{cases} \quad Z_1(k\gamma_j x) = \begin{cases}\sin(k\gamma_j x) \text{ if } \epsilon_j^{(i)} = 1 \\ \exp(k\gamma_j x) \text{ if } \epsilon_j^{(i)} = -1\end{cases} \quad j = (s,d)$$

$$W_0(k_j x) = \begin{cases}\sin(k\gamma_j x) \\ \exp(-k\gamma_j x)\end{cases} \quad W_1(k\gamma_j x) = \begin{cases}-\cos(k\gamma_j x) \text{ if } \epsilon_j^{(i)} = 1 \\ \exp(-k\gamma_j x) \text{ if } \epsilon_j^{(i)} = -1\end{cases} \quad \gamma_s = \alpha^{(i)},\ \gamma_d = \beta^{(i)} \quad (4)$$

$v = \omega/k$ is the phase velocity; ω is the circular frequency; k is the wave number in the axial direction. $v_s^{(i)}$, $v_d^{(i)}$ are the phase velocity of shear and pressure waves, respectively, in an infinite isotropic elastic body made of the material of the ith layer; $\rho_i^{(i)}$ and $\mu_i^{(i)}$ are the density and the shear modulus of the ith layer, respectively.

For a plate made of linear isotropic elastic layers the components of stress for each layer may be obtained from Eq. (1) and may be substituted into the boundary and interface conditions resulting in a set of eighteen homogeneous, linear algebraic equations in terms of the eighteen constants $A_j^{(i)}$, $B_j^{(i)}(i,j = 1,2,3)$. For a nontrivial solution, the determinant of their coefficients must vanish.

The six constants $A_3^{(i)}$, $B_3^{(i)}(i = 1,2,3)$ appear only in six of these equations whereas the remaining twelve constants $A_j^{(i)}$, $B_j^{(i)}$ $(j = 1,2;\ i = 1,2,3)$ appear in the twelve remaining equations. Consequently, two independent conditions exist for determining nontrivial solutions of the equations of motion; namely

$$D_1 = |\bar{C}_{ij}| = 0(i,j = 1,2,\ldots,6) \quad (5)$$

and

$$D_2 = |C_{ij}| = 0(i,j = 1,2,\ldots,12) \quad (6)$$

Equation (5) is the frequency equation for face-shear (SH) motion in the plate ($u_x = u_z = 0$) whereas Eq. (6) is the frequency equation for coupled longitudinal and flexural (p and SV) motion ($u_y = 0$). In general if the cross section of the plate is not symmetric about the $x = 0$ plane, the face-shear and the longitudinal-flexural motions do not degenerate into symmetric and antisymmetric families of modes.

The elements of the determinant D_1 are given by

$$\bar{C}_{11} = \epsilon_s^{(1)}\mu^{(1)}k\alpha^{(1)}Z_1(k\alpha^{(1)}x_1), \quad \bar{C}_{12} = \mu^{(1)}k\alpha^{(1)}W_1(k\alpha^{(1)}x_1)$$

$$\bar{C}_{21},\bar{C}_{22} = \bar{C}_{11},\bar{C}_{12} \quad \text{with } x_2 \text{ substituted for } x_1 \quad (7)$$

$$\bar{C}_{23},\bar{C}_{24} = -\bar{C}_{11},-\bar{C}_{12}$$

with superscript (1) changed to (2) and x_2 substituted for x_1

$$\bar{C}_{31} = -Z_0(k\alpha^{(1)}x_2), \quad \bar{C}_{32} = -W_0(k\alpha^{(1)}x_2)$$

$$\bar{C}_{33},\bar{C}_{34} = -\bar{C}_{31},-\bar{C}_{32} \quad \text{with superscript (1) changing to (2)}$$

$$\bar{C}_{43},\bar{C}_{44},\bar{C}_{53},\bar{C}_{54} = -\bar{C}_{31},-\bar{C}_{32},-\bar{C}_{11},-\bar{C}_{12} \quad \text{with } (1)\rightarrow(2) \text{ and } x_1 \rightarrow -x_2$$

$$\bar{C}_{45},\bar{C}_{46},\bar{C}_{55},\bar{C}_{56} = \bar{C}_{31},\bar{C}_{32},\bar{C}_{11},\bar{C}_{12} \quad \text{with } (1) \rightarrow (3) \text{ and } x_1 \rightarrow -x^2$$

$$\bar{C}_{65},\bar{C}_{66} = \bar{C}_{11},\bar{C}_{12} \quad \text{with } (1)\rightarrow(3) \text{ and } x_1 \rightarrow x_3$$

The elements of the determinant D_2 are listed below

$$C_{11} = -\mu^{(1)}k^2(\epsilon_s^{(1)}\alpha^{(1)2} - 1)Z_0(k\beta^{(1)}x_1)$$

$$C_{12} = -\mu^{(1)}k^2(\epsilon_s^{(1)}\alpha^{(1)2} - 1)W_0(k\beta^{(1)}x_1)$$

$$C_{13} = -2\mu^{(1)}\epsilon_s^{(1)}k^2\alpha^{(1)}Z_1(k\alpha^{(1)}x_1)$$

$$C_{14} = -2\mu^{(1)}k^2\alpha^{(1)}W_1(k\alpha^{(1)}x_1)$$

$$C_{21} = 2\mu\epsilon_d^{(1)}\beta^{(1)}Z_1(k\beta^{(1)}x_1), \quad C_{22} = 2\mu^{(1)}k^2\beta^{(1)}W_1(k\beta^{(1)}x_1)$$

$$C_{23} = -\mu^{(1)}k^2(\epsilon_s^{(1)}\alpha^{(1)2} - 1)Z_0(k\alpha^{(1)}x_1)$$

$$C_{24} = -\mu^{(1)}k^2(\epsilon_s^{(1)}\alpha^{(1)2} - 1)W_0(k\alpha^{(1)}x_1)$$

$$C_{3i},C_{4i}(i = 1,2,3,4) = C_{2i},C_{1i}(i = 1,2,3,4) \quad \text{with } x_2 \text{ substituted for } x_1$$

$$C_{3i},C_{4i}(i = 5,6,7,8) = -C_{2i},-C_{1i}(i = 1,2,3,4) \text{ with } x_1 \rightarrow x_2$$

$$C_{51} = -\epsilon_d^{(1)}k\beta^{(1)}Z_1(k\beta^{(1)}x_2), \quad C_{52} = -k\beta^{(1)}W_1(k\beta^{(1)}x_2)$$

$$C_{53} = -kZ_0(k\alpha^{(1)}x_2), \quad C_{54} = -kW_0(k\alpha^{(1)}x_2)$$

$$C_{61} = -kZ_0(k\beta^{(1)}x_2), \quad C_{62} = -kW_0(k\beta^{(1)}x_2)$$

$$C_{63} = -\epsilon_s^{(1)}k\alpha^{(1)}Z_1(k\alpha^{(1)}x_2), \quad C_{64} = -k\alpha^{(1)}W_1(k\alpha^{(1)}x_2)$$

$$C_{5i},C_{6i}(i = 5,6,7,8) = -C_{5i},-C_{6i}(i = 1,2,3,4) \quad \text{with superscript (1) changed to (2)}$$

$$C_{7i},C_{8i}(i = 5,6,7,8) = -C_{6i},-C_{5i}(i = 1,2,3,4) \quad \text{with } (1) \rightarrow (2)\ x_2 \rightarrow -x_2$$

$$C_{7i},C_{8i}(i = 9,10,11,12) = C_{6i},C_{5i}(i = 1,2,3,4) \quad \text{with } (1) \rightarrow (3) \text{ and } x_2 \rightarrow -x_2$$

$$C_{9i},C_{10,i}(i = 5,6,7,8) = -C_{1i},-C_{2i}(i = 1,2,3,4) \quad \text{with } (1) \rightarrow (2) \text{ and } x_2 \rightarrow -x_2$$

$$C_{9i},C_{10,i}(i = 9,10,11,12) = C_{2i},C_{1i}(i = 1,2,3,4) \quad \text{with } (1) \rightarrow (3) \text{ and } x_2 \rightarrow -x_2$$

$$C_{11,i},C_{12,i}(i = 9,10,11,12) = C_{2i},C_{1i}(i = 1,2,3,4) \quad \text{with } (1) \rightarrow (3) \text{ and } x_2 \rightarrow x_3$$

where

$$x_1 = -\left(\frac{h^{(2)}}{2} + h^{(1)}\right) x_2 = -\frac{h^2}{2} \quad x_3 = \frac{h^{(2)}}{2} + h^{(3)}$$

If the materials of Eqs. (1) and (2) or Eqs. (2) and (3) are identical, the frequency Eqs. (5) and (6) reduce to those for motion in a two-layered plate obtained by Jones.[2] If all the layers of the plate are composed of the same material, each of the Eqs. (5) and (6) degenerates into the product of two determinants. One for symmetric and the other is for antisymmetric motion in a homogeneous plate, discussed by Mindlin[3] and Meeker and Meitzler.[4]

When the wave number k vanishes, the motion becomes independent of the axial coordinate. A number of the terms of the frequency determinants vanish and the frequency equation for longitudinal-flexural motion $D_2 = 0$ may be written as the product of two determinants. One for thickness shear modes ($u_x = u_y = 0$) and the other for thickness stretch modes ($u_y = u_z = 0$).

If the material and thickness of the layers (1) and (3) are identical, Eqs. (5) and (6) degenerate into the product of two determinants

$$\begin{aligned} D_1 &= 2D_1{}^S D_1{}^A = 0 \\ D_2 &= 2D_2{}^S D_2{}^A = 0 \end{aligned} \tag{8}$$

where

$$D_1{}^S = \begin{vmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} \end{vmatrix}, \quad D_1{}^A = \begin{vmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{24} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{34} \end{vmatrix} \tag{9}$$

$$D_2{}^S = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{38} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{48} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{58} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{68} \end{vmatrix}$$

$$D_2{}^A = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{36} & C_{37} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{46} & C_{47} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{56} & C_{57} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{66} & C_{67} \end{vmatrix} \tag{10}$$

$D_1{}^S = 0$ and $D_1{}^A = 1$ represent face-shear motion, symmetric and antisymmetric with respect to the middle plane ($x = 0$), respectively. Whereas, $D_2{}^S = 0$ and $D_2{}^A = 0$ represent longitudinal (symmetric) and flexural (antisymmetric) motion, respectively. The frequency equations for longitudinal and flexural motion ($D_2{}^S = 0$, $D_2{}^A = 0$) were obtained by Y. Y. Yu[5,6] and were evaluated numerically in the range of small frequencies and large wavelengths.

Numerical Results

For given geometric and physical parameters the frequency Eqs. (5) and (6) constitute transcendental relationships between the nondimensionalized wave number $\zeta = h^{(2)}/L = kh^{(2)}/\pi$ and the nondimensionalized frequency $\Omega = \omega h^{(2)} [\rho^{(2)}]^{1/2}/\pi[\mu^{(2)}]^{1/2}$.

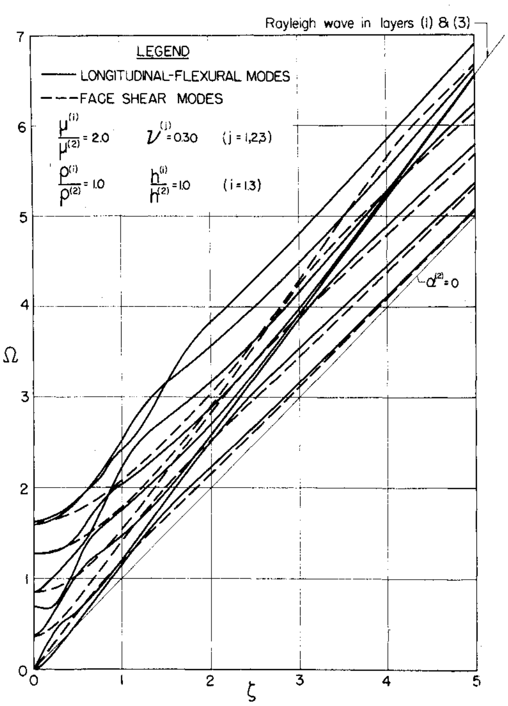

Fig. 2 Frequency spectrum for a symmetric plate.

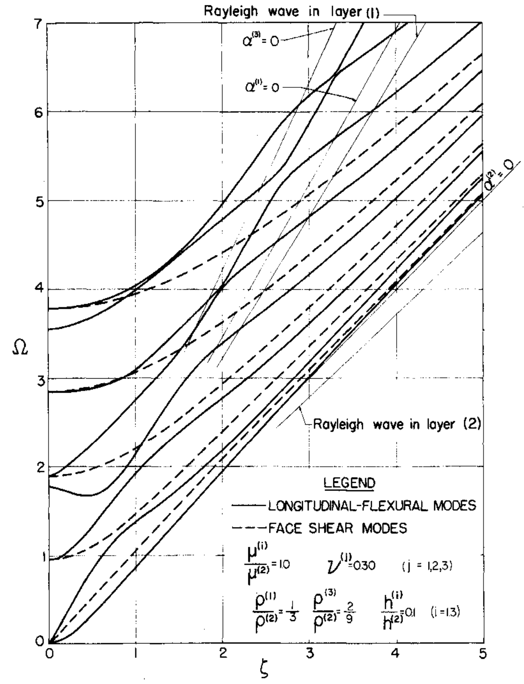

Fig. 3 Frequency spectrum for a nonsymmetric plate.

A procedure analogous to that described in Ref. 7 has been programmed on an IBM 7044/7094 direct coupled system for numerical evaluation of the frequency equations.

The frequency spectrum for symmetric, three-layered plates is shown in Fig. 2. In this case, the face-shear modes and the longitudinal-flexural modes involve uncoupled symmetric or antisymmetric motion. The first dashed line is the frequency line of the zeroth symmetric face-shear mode. In plates made of one material, this mode is nondispersive; its velocity does not vary with the wave length. In three-layered plates, this mode could be considerably dispersive, as shown in Fig. 2. The higher face-shear modes are alternately, one antisymmetric and one symmetric.

The lowest longitudinal-flexural mode (full-line) is antisymmetric (flexural) while the next (full-line) is symmetric (longitudinal).

The frequency spectrum for nonsymmetric three-layered plates is shown in Fig. 3. In this case the face-shear modes and the longitudinal flexural modes do not uncouple into symmetric and antisymmetric motion.

In Fig. 2, the physical parameters of the plate are such that $v_R{}^{(1)} = v_R{}^{(3)} > v_s{}^{(2)}$, while in Fig. 3, the physical parameters of the plate are such that the lowest shear-wave velocity is $v_s{}^{(2)}$ and $v_R{}^{(i)} > v_s{}^{(2)}$ ($i = 1,3$). Where $v_R{}^{(i)}$ is the velocity of propagation of Rayleigh surface waves in the material of the ith layer. Consequently, as has been established analytically[7] at large wave numbers, the phase velocity of all longitudinal-flexural modes approaches asymptotically that of the shear waves in layer 2. As apparent in Fig. 2, the frequency lines of higher modes terrace on the frequency line for Rayleigh waves in the outer and inner layers, prior to turning to the $\alpha^{(2)} = 0$ direction.

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Linear System Stability via Liapunov's Direct Method and the Square Integral

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Introduction

An application of Liapunov's second method to autonomous systems described by linear differential equations of the form

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = 0 \tag{1}$$

or by linear difference equations of the form

$$\sum_{i=0}^{n} a_i y(m+i) = 0 \qquad (m = \ldots -1, 0, 1, \ldots) \tag{2}$$

results in a set of necessary and sufficient conditions for asymptotic stability in the large.[1,2] These conditions must be equivalent, respectively, to the Routh stability criterion and the analogous criterion for discrete time systems, referred to as the table form criterion.[3–5] These equivalences have been shown by a number of authors.[6–11]

In this paper, the equivalence is established for both continuous and discrete time systems in completely parallel developments. This is accomplished by deriving and proving the stability criteria directly from appropriate Liapunov functions obtained in a very natural way.

The derivations begin with the square integral and summation defined, respectively, as

$$V = \int_t^\infty y^2(\tau)d\tau$$

for continuous time systems and by

$$V = \sum_{k=m}^{\infty} y^2(k)$$

crete time systems. The convolution integrals from transform theory along with residues are used to express the square integral and summation as quadratic forms in the state variables

$$V(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{x}.$$

These forms are shown to be Liapunov functions for the systems.

The Routh criterion and the Routh-type criterion are then derived and proven to be necessary and sufficient conditions for the respective quadratic forms to be Liapunov functions and, therefore, for the systems to be asymptotically stable in the large.

Continuous Time Systems

A. *Derivation of the Liapunov function*

Consider the phase variable representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t), \quad y(t) = x_1(t) \tag{3}$$

of a system governed by the linear, constant coefficient differential equation given in Eq. (1).

Assume that the system is asymptotically stable in the large (ASL). Then, the square integral V defined by

$$V = \int_t^\infty x_1^2(\tau)d\tau$$

exists and can be evaluated using the convolution integral of the Laplace transform and residue theory (Ref. 12, Ref. 13, pp. 10–15) as the quadratic form

$$V(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{x} \tag{4}$$

where (for n odd)

$$\mathbf{B} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_n & 0 & \ldots & 0 \end{bmatrix},$$

$$\mathbf{Q}_n = \begin{bmatrix} N_1 & 0 & N_2 & \ldots & N_{(n+1)/2} \\ 0 & -N_2 & 0 & \ldots & 0 \\ N_2 & 0 & N_3 & \ldots & N_{(n+3)/2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ N_{(n+1)/2} & 0 & N_{(n+3)/2} & \ldots & N_n \end{bmatrix} \tag{5}$$

and the set of factors N_i is the solution of the set of equations

$$\begin{bmatrix} a_0 & a_2 & \ldots & a_{n-1} & 0 & \ldots & 0 \\ 0 & a_1 & \ldots & a_{n-2} & a_n & \ldots & 0 \\ 0 & a_0 & \ldots & a_{n-3} & a_{n-1} & \ldots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & 0 & \ldots & a_0 & a_2 & \ldots & a_{n-1} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \cdot \\ \cdot \\ \cdot \\ N_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1/a_n \end{bmatrix} \tag{6}$$

It is now shown that $V(\mathbf{x})$ is a Liapunov function for the system of Eq. (1). The integrand $(x_1)^2$ is non-negative. If $\mathbf{x} = \mathbf{0}$ at time t, it can be seen from Eq. (3) that x_1 and its first $n-1$ derivatives are zero at time t, and consequently that $x_1 = 0$ in the interval $[t, \infty]$. Therefore, $V = 0$. Also, if $V = 0$, then $x_1 = 0$ in that interval and $\mathbf{x} = \mathbf{0}$. It follows that V is positive definite and so is $V(\mathbf{x})$ as given in Eq. (4).

The derivative of V with respect to time given by

$$\dot{V} = d/dt\left[\int_t^\infty x_1^2(\tau)d\tau\right] = -x_1^2(t) \tag{7}$$

is negative semidefinite. Also, from the above, $\dot{V}$ is not identically zero anywhere other than $\mathbf{x} = \mathbf{0}$.

Using the linearity of the system with the above, it is seen that $V(\mathbf{x})$ is a Liapunov function for the system if it is ASL. Applying Liapunov's theorem, it is concluded that the system is ASL if and only if $V(\mathbf{x})$ is a Liapunov function.

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